

# Anderson transitions in three-dimensional disordered systems with randomly varying magnetic flux

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(February 1, 2008)*

The Anderson transition in three dimensions in a randomly varying magnetic flux is investigated in detail by means of the transfer matrix method with high accuracy. Both, systems with and without an additional random scalar potential are considered. We find a critical exponent of  $\nu = 1.45 \pm 0.09$  with random scalar potential. Without it,  $\nu$  is smaller but increases with the system size and extrapolates within the error bars to a value close to the above. The present results support the conventional classification of universality classes due to symmetry.

71.30.+h, 71.23.-k, 72.15.Rn, 64.60.-i

Since the pioneering work of Anderson [1], the disorder-induced metal-insulator transition, which is one of the most fundamental quantum phase transitions in condensed matter physics, has attracted considerable attention [2,3]. Depending on the symmetry, the critical behavior of this Anderson transition(AT) is conventionally classified into three universality classes: the orthogonal, the unitary and the symplectic class [4]. Systems invariant under spin rotation as well as under time reversal belong to the orthogonal class. Unitary systems are characterized by the absence of time-reversal symmetry, due to, for instance, a magnetic field. Systems without spin rotation invariance belong to the symplectic class.

The AT in a homogeneous magnetic field has been studied extensively for many years, mainly in connection with the quantum Hall effect [5]. In two dimensions(2D) in the presence of a strong magnetic field, the AT is marginal. States at the centers of the Landau bands are critical and all the other are localized. At the band centers, the localization length diverges with the exponent  $\nu \sim 2.4$  [6]. In 3D, there exist extended states and the AT takes place [7–9]. The latter has recently been re-analyzed and the critical exponent for the localization length has been determined to be  $1.43 \pm 0.06$  [10].

When the magnetic field is uniform in space, randomness is introduced by a random scalar potential. On the other hand, in recent years, there has also been consider-

able interest in 2D systems subject to a spatially random magnetic field, mainly in connection with the fractional quantum Hall effect [11]. The random magnetic field introduces randomness as well as the absence of invariance under time reversal in a system.

In 3D, the AT in the presence of a random vector potential and without a random scalar potential, has been investigated numerically. The data suggested [12] that the mobility edge is very close to the band edge. The exponent for the localization length has been estimated to be  $\nu \approx 1$  [12] which is considerably smaller than that in the case with an additional random scalar potential and in a uniform magnetic field. It has also been reported [8,13] that in the presence of a random scalar potential, the critical exponent has a universal value, irrespective of whether the magnetic field is uniform or random. This seemed to indicate that the AT in a random vector potential but without random scalar potential is different from the one with a random scalar potential. It should be noted that this would question the validity of the above conventional classification of the AT because in both cases the time reversal symmetry is broken and hence these two systems should belong to the same, namely the unitary universality class.

The critical exponent  $\nu \approx 1$  for the 3D system with a random magnetic field has been obtained by the finite-size scaling method [14]. This method has been applied successfully to analyze the critical behavior of the AT [3]. In most cases, however, the numerical analyses have been restricted to energies near the band center. It has been reported that systematic scaling behavior has not been clearly observed for energies away from the band center [15,16]. For the 3D system with a random magnetic field, the mobility edge lies quite close to the effective band edge [12]. It is therefore imperative to investigate the present problem with considerably higher accuracy and to examine carefully whether or not the scaling behavior is modified by adding a random scalar potential.

Recently, high-accuracy scaling analyses of the Anderson transition have been performed by several authors

[10,17]. It has been concluded [10] that, by reducing the errors of raw data to  $0.1 \sim 0.2\%$ , one can numerically distinguish the unitary from the orthogonal class, which was impossible when only low accuracy data with  $\sim 1\%$  accuracy were used.

Encouraged by this recent success, we have started a numerical high-precision finite size scaling project, in order to clarify the above mentioned discrepancy between the critical exponent of the AT far away from the band center induced solely by randomness in a vector potential and the exponent obtained for other unitary systems.

We found a clear systematic dependence of the exponent on the system-size in the former case which would introduce corrections to scaling. These would become smaller than the statistical error only for system sizes larger than those which are presently achievable. We estimate the asymptotic limit for the exponent to be  $\nu \sim 1.4$ , when the transition is near the band edge. This behavior is not changed significantly when shifting the mobility edge by adding a weak random scalar potential.

The model is defined by the Hamiltonian [12]

$$H = t \sum_{\langle i,j \rangle} \exp(i\theta_{i,j}) C_i^\dagger C_j + \sum_i V_i C_i^\dagger C_i, \quad (1)$$

where  $C_i^\dagger (C_i)$  denotes the creation(annihilation) operator of an electron at the site  $i$  of a 3D cubic lattice. Energies  $\{V_i\}$  denote the random scalar potential distributed independently and uniformly in the range  $[-W/2, W/2]$ . The Peierls phase factors  $\exp(i\theta_{i,j})$  describe a random vector potential or magnetic field. We confine ourselves to phases  $\{\theta_{i,j}\}$  which are distributed independently and uniformly in  $[-\pi, \pi]$ . The hopping amplitude  $t$  is assumed to be the energy unit,  $t = 1$ .

We consider quasi-1D systems with cross section  $M \times M$  [14,17]. The Schrödinger equation  $H\psi = E\psi$  in such a bar-shaped system can be rewritten using transfer matrices  $T_n (2M^2 \times 2M^2)$

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = T_n \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}, \quad T_n = \begin{pmatrix} E - H_n & -I \\ I & 0 \end{pmatrix} \quad (2)$$

( $n = 1, 2, \dots$ ) where  $\psi_n$  and  $H_n$  denote the set of coefficients of the state  $\psi$  and the Hamiltonian of the  $n$ -th slice, respectively. The identity matrix is denoted by  $I$ . The off-diagonal parts of the transfer matrix  $T_n$  can be expressed by the identity matrix because the phases in the transfer-direction can be removed by a gauge transformation [12]. The logarithms of the eigenvalues of the limiting matrix  $T$

$$T \equiv \lim_{n \rightarrow \infty} \left[ \left( \prod_{i=1}^n T_i \right)^\dagger \left( \prod_{i=1}^n T_i \right) \right]^{1/2n} \quad (3)$$

are called the Lyapunov exponents. The smallest Lyapunov exponent  $\lambda_M$  along the bar is estimated by a

technique which uses the product of these transfer matrices [3,14]. The relative accuracies for the smallest Lyapunov exponents achieved here are  $0.2\%$  for  $M \leq 10$  and  $0.25\% \sim 0.3\%$  for  $M = 12$ . The localization length  $\xi_M$  along the bar is given by the inverse of the smallest Lyapunov exponent,  $\xi_M = 1/\lambda_M$ .

The assumption of one-parameter scaling for the renormalized localization length  $\Lambda_M \equiv \xi_M/M$  implies

$$\Lambda_M = f(\xi/M), \quad (4)$$

where  $\xi = \xi(E, W)$  is the relevant length scale in the limit  $M \rightarrow \infty$  [14]. Near the mobility edge  $E_c(W)$ ,  $\xi$  diverges with an exponent  $\nu$  as  $\xi \sim x^{-\nu}$  with  $x = (E - E_c)/E_c$ . If the transition is driven by the disorder  $W$  at a constant energy,  $x = (W_c - W)/W_c$ . At the mobility edge,  $\Lambda_M$  becomes scale-invariant. The quantity  $\Lambda_M$  is a smooth function of  $E$  and  $W$ , and we can expand it as a function of  $x$  as

$$\Lambda_M = \Lambda_c + \sum_{n=1}^{\infty} A_n (M^{1/\nu} x)^n. \quad (5)$$

By fitting our data to the above function, we can determine the critical exponent  $\nu$  and the mobility edge accurately. In practice, we truncated the series (5) at the third order.

We used the standard  $\chi^2$ -fitting procedure [18]. In order to check the goodness of the fit, we also evaluated the probability  $Q$  that the  $\chi^2$  will exceed the minimum value  $\chi^2_{\min}$  actually obtained by the fit. The probability  $Q$  is evaluated via the incomplete gamma functions and the condition  $Q > 0.001$  is often regarded as an acceptable condition for the fitting function [18]. If the value of  $Q$  is too small, in other words, the minimum value of the  $\chi^2$  is considerably large, there may be systematic deviations of the numerical data from the fitting function. In the recent work [10] on the Anderson transition in 3D orthogonal and unitary systems, it has been demonstrated that the above fitting function up to the third order is in fact valid. The error bars are estimated by using the Hessian matrix and the confidence interval is chosen to be 95.4%.

We consider first the AT at the band center  $E = 0$  in the presence of a strong random scalar potential as well as random vector potential. The renormalized localization lengths  $\Lambda_M$  evaluated for the disorder  $W$  in the range [17.8, 19.8] and sizes  $M = 6, 8, 10$  and 12 are shown in figure 1. The above described fit yields the critical exponent  $\nu = 1.45 \pm 0.09$  and the critical disorder  $W_c = 18.80 \pm 0.04$ . The renormalized localization length  $\Lambda_c$  at the critical point is  $0.558 \pm 0.003$ . The value of  $Q$  is  $\sim 0.89$  which confirms the validity of the fitting function (5) and thus of the one-parameter scaling behavior of  $\Lambda_M$  in this range of the disorder  $W$ . The error bars of the present results are at least a factor of 3 smaller than those of the previous estimates [8].

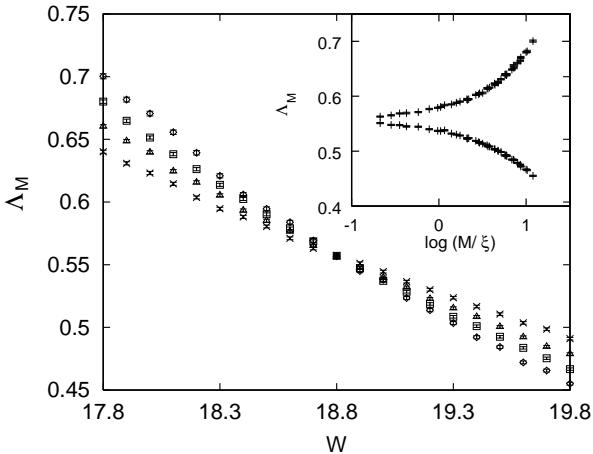


FIG. 1. The renormalized localization length  $\Lambda_M$  as a function of disorder  $W$  for different sizes. The crosses, triangles, the squares and the diamonds correspond to  $M = 6, 8, 10$  and  $12$ , respectively. Inset: The scaling function.

Next, we concentrate on the AT in the random magnetic field *but* without random scalar potential ( $W = 0$ ). In the previous work [12], the energy range used for the scaling analysis was assumed to be  $4.3 \leq |E| \leq 4.5$ . This is very close to the band edge and thus the density of states(DOS) is rapidly decreasing [12]. To get rid of the influence by this rapid change of the DOS, the energy window for the scaling analysis should be taken to be as small as possible. We therefore choose calculated data for  $4.39 \leq E \leq 4.44$ . The energy window is 4 times smaller than that in [12]. The numerical data for  $\Lambda_M$  are shown in figure 2. The transition can be located around  $E \approx 4.415$ . We then fit the data for  $M = 6, 8, 10$  and  $12$  to the function (5). This yields  $E_c \approx 4.414$ ,  $\nu \approx 1.2$  and  $\Lambda_c \approx 0.51$ , which is consistent with the previous estimates [12]. It is found, however, that in the present case, the value of  $Q$  turns out to be very small , namely  $\sim 10^{-14}$ , in contrast to the above considered case of the band center. This shows that systematic deviations of the data from the fitting function are very likely to exist. We therefore have to analyze the numerical data much more carefully.

In order to get insight into the origin of the deviations of the data from the fitting function, we carried out the fits using different combinations of system-sizes  $M_1$  and  $M_2 = M_1 + 2$  for  $M_1 = 6, 8$  and  $10$  (table I). Although the crossing point is almost size independent, the exponent shows a systematic dependence on the system cross section. For  $M = 6$  and  $8$ , in particular, the exponent is close to 1, while for the other two cases it is around 1.3 which is close to the value estimated for a system in a uniform magnetic field. In addition, for  $M = 6$  and  $8$  the value of  $Q$  becomes much smaller than those for the other two cases, which could imply that the fitting function is not working well for these smaller sizes.

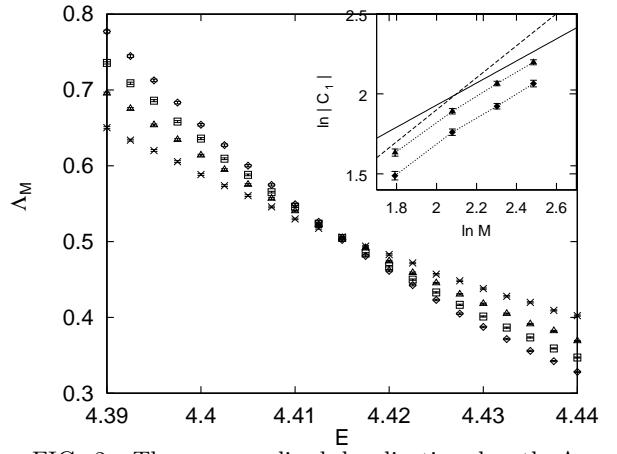


FIG. 2. The renormalized localization length  $\Lambda_M$  as a function of energy  $E$  for  $W = 0$ . The crosses, triangles, the squares and the diamonds correspond to  $M = 6, 8, 10$  and  $12$ , respectively. Inset: The logarithm of the derivative  $C_1 \equiv d\Lambda_M/dE$  for  $W = 0$  at  $E = 4.414 \approx E_c$ (filled triangles) and for  $W = 1$  at  $E = 4.451 \approx E_c$ (filled diamonds) as a function of  $\ln M$ . The solid line and the dashed lines represent the slope with  $\nu = 1.45$  and  $\nu = 1$ , respectively.

We also analyzed the data in a different way. We performed a third order polynomial fit for each size and estimated the derivative  $C_1 \equiv d\Lambda_M/dE|_{E=E_c}$  at the critical energy  $E_c = 4.414$ . The value of  $Q$  in this case is larger than 0.1 for any size, which means that the 3rd order polynomial fit itself is working fairly well. We plot  $\ln |C_1|$  as a function of  $\ln M$  (figure 2, inset). If scaling works, the slope is related to the exponent by  $1/\nu$ . It is clear from figure 2 (inset) that the slopes deviate from 1 and are likely to approach the value  $\nu \sim 1.45$ , when  $M$  is increased. These two analyses suggest that the present system-sizes may not be large enough to observe clear one-parameter scaling behavior for  $W = 0$ .

To see whether these features are specific to  $W = 0$ , we performed the same analysis for a system with weak scalar randomness  $W = 1$ . The results of the fits using different combinations of system-sizes and by the third order polynomial fits are listed in table II and shown in figure 2 inset, respectively. In table II, we see that the exponent for  $M = 6$  and  $8$  again turns out to be close to 1 and deviates from those for the other two cases, although the crossing points are rather stable with respect to the change of sizes. It should also be noted that the value of  $\Lambda_c$  is almost the same as for  $W = 0$ . In the inset of figure 2, we can observe again the deviation of the slopes from 1 as the size is increased. Thus, the system with a weak random scalar potential shows similar behavior as that without a random scalar potential.

The results for  $W = 1$  and  $W = 0$  indicate that the correction to scaling is not specific to the case  $W = 0$ . It is natural to expect that this is due to the fact that the transition is near the band edge. In both cases,  $W = 1$  and  $W = 0$ , the fits using the size  $M = 6$  give smaller values

of the critical exponent  $\nu$ . We conclude that this is the reason why the exponent found in the previous work [12] was smaller. The present analysis for  $W = 1$  and  $W = 0$  also shows that finite-size corrections to scaling exist for the presently achievable system sizes ( $6 \leq M \leq 12$ ), which might be the reason of the discrepancy between  $\Lambda_c$  near the band edge and that at the band center.

In summary, we have re-investigated in detail the Anderson transition in the random magnetic field with and without random scalar potential. We have evaluated the localization length along quasi-1D systems with high accuracy and examined the scaling behavior of the renormalized localization length near the transition. We have confirmed the one-parameter scaling behavior for the transition at the band center with a relatively strong random scalar potential and found the exponent  $1.45 \pm 0.09$ . This value agrees well with the recent precise results for systems in a uniform magnetic field [10]. We have also performed the finite-size scaling analysis for both, the system without random scalar potential ( $W = 0$ ) and with a weak random scalar potential ( $W=1$ ). In both cases, deviations of numerical data from the scaling ansatz are found, especially for smaller sizes. As the size is increased, the exponent is more likely to approach to a value around 1.4 rather than the value of 1. In particular, no evidence for the exponent 1 has been found.

On the basis of the present results, we conclude that there exists no evidence that the critical behavior in a 3D system in a random magnetic field is different from that for other unitary systems. This supports the conventional classification of the AT by universality classes due to symmetry.

The authors thank M. Batsch, A. MacKinnon and I. Zharekeshev for valuable discussions. The numerical calculations were done on a FACOM VPP500 of Institute for Solid State Physics, University of Tokyo and in computer facilities of I. Institut für Theoretische Physik, Universität Hamburg. This work was supported in part by the EU-project FHRX-CT96-0042 and by the Deutsche Forschungsgemeinschaft via Project Kr627/10 and the Graduiertenkolleg “Nanostrukturierte Festkörper”. One of the authors (T.K.) thanks Alexander von Humboldt Foundation for financial support during his stay at University of Hamburg.

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$(M_1, M_2)$	$\Lambda_c$	$\nu$	$E_c$	$Q$
(6, 8)	$0.514 \pm 0.005$	$1.05 \pm 0.07$	$4.414 \pm 0.001$	$\sim 10^{-5}$
(8, 10)	$0.516 \pm 0.007$	$1.26 \pm 0.09$	$4.414 \pm 0.001$	$\sim 0.89$
(10, 12)	$0.51 \pm 0.01$	$1.32 \pm 0.12$	$4.414 \pm 0.001$	$\sim 0.99$

TABLE I. Results of the fits for different sizes in the absence of random scalar potential ( $W = 0$ ).

$(M_1, M_2)$	$\Lambda_c$	$\nu$	$E_c$	$Q$
(6, 8)	$0.510 \pm 0.005$	$1.09 \pm 0.08$	$4.451 \pm 0.001$	$\sim 0.91$
(8, 10)	$0.519 \pm 0.008$	$1.36 \pm 0.12$	$4.450 \pm 0.001$	$\sim 0.56$
(10, 12)	$0.51 \pm 0.01$	$1.34 \pm 0.14$	$4.452 \pm 0.002$	$\sim 0.95$

TABLE II. Results of the fits for different sizes in the presence of the weak random scalar potential ( $W = 1$ ). The energy window is taken to be  $4.425 \leq E \leq 4.475$  and the same number of data points as the case of  $W = 0$  are used for the scaling analysis.

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